1. (30 points) Use your calculator to evaluate the integrals for these application problems.

(a) Find the volume of the solid generated by rotating the region bounded by the graphs of \( y = x, \ y = xe^x, \) and \( x = 1 \) around the \( y \)-axis.

\[
V = \int_0^1 2\pi x (xe^x - x) \, dx
\]

(b) A house has an oil tank in the basement in the shape of a cylinder lying on its side. It has to be emptied for repair. The radius of the cylinder is 3 ft and the length is 8 ft. It is half full of heating oil. If the density of heating oil is 63 lbs per cubic foot, find the work required to pump the contents to an oil truck that is 12 feet higher than the top of the tank.

Find the centroid of the region bounded by the graphs of \( y = \cos x, \ y = 0, \) and \( x = \pi/4. \)
2. (12 points) Solve the initial value problem \( \frac{dy}{dx} - \frac{2y}{x} = 1 - x, \ y(1) = 2 \).

\[
\frac{dy}{dx} = x^2 - x^{-1}
\]

\[
\int (x^2 - x^{-1}) \, dx = \frac{x^3}{3} - x + C
\]

\[
y = \frac{x^3}{3} - x + C
\]

3. (48 points) Do any 4 of the following 5 integrals.

(a) \( \int \cos^3 x \sin^5 x \, dx \)

\[
u = \cos x \quad \Rightarrow \quad du = -\sin x \, dx
\]

\[
\sin^2 x = 1 - \cos^2 x
\]

\[
\sin^4 x = (1 - \cos^2 x)^2
\]

\[
\int \cos^3 x \sin^5 x \, dx = \int u^2 (1-u^2)^2 \, du
\]

\[
= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C
\]

(b) \( \int_0^1 \frac{x^3}{\sqrt{9 - x^2}} \, dx \)

\[
u = 9 - x^2 \quad \Rightarrow \quad dv = -2x \, dx
\]

\[
\int_0^1 \frac{x^3}{\sqrt{9 - x^2}} \, dx = \int \frac{x^3}{\sqrt{9 - x^2}} \, dx
\]

\[
= \frac{x^2}{\sqrt{9 - x^2}} \cdot (-\frac{1}{2}) \, du
\]

\[
= -\frac{1}{2} \int \frac{9 - u^2}{u} \, du
\]

\[
= -\frac{1}{2} \left( 9u^{1/2} - u^{3/2} \right) + C
\]

\[
= \frac{18 \sqrt{2} - 38\sqrt{2}}{3}
\]

(c) \( \int \cos 7x \cos x \, dx \)

\[
= \frac{1}{2} \int \cos (7x-x) + \frac{1}{2} \cos (7x+x) \, dx
\]

\[
= \frac{1}{2} \left( \frac{\sin (6x)}{6} + \frac{\sin (8x)}{8} \right) + C
\]
4. (12 points) Find \( \int \frac{x^2 + x - 1}{x - x^3} \, dx \). You may use your calculator only to solve equations for the partial fractions decomposition (not to do the entire decomposition or any resulting integrals).

\[
\frac{x^2 + x - 1}{x - x^3} = \frac{x^2 - x - 1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}
\]

So \( x^2 + x - 1 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x) \)

\[
\begin{align*}
\text{plug in } x = 0: & \quad -1 = A \\
\text{plug in } x = -1: & \quad -1 = C(-1)(2) \quad \text{so } C = \frac{1}{2} \\
\text{plug in } x = 1: & \quad 1 = B(1)(2) \quad \text{so } B = \frac{1}{2}
\end{align*}
\]

\[
\int \frac{x^2 + x - 1}{x - x^3} \, dx = \int \left( \frac{-1}{x} + \frac{1}{1-x} + \frac{1}{1+x} \right) \, dx
\]

\[
= \left[ -\ln|x| - \frac{1}{2} \ln|1-x| + \frac{1}{2} \ln|1+x| \right] + C
\]
5. (12 points) Use the trapezoidal rule to approximate \( \int_1^2 \frac{1}{2+x} \, dx \) using 6 subdivisions. Use your calculator to give a decimal approximation.

\[
\Delta x = \frac{2-1}{6} = \frac{1}{6}
\]

\[
\int_1^2 \frac{1}{2+x} \, dx \approx \frac{1}{2} \left( f(1) + 2 f \left( \frac{7}{6} \right) + 2 f \left( \frac{8}{6} \right) + 2 f \left( \frac{9}{6} \right) + 2 f \left( \frac{10}{6} \right) + f \left( \frac{11}{6} \right) \right)
\]

\[
= \frac{1}{12} \left( 1 + 2 \cdot \frac{6}{9} + 2 \cdot \frac{6}{20} + 2 \cdot \frac{6}{21} + 2 \cdot \frac{6}{22} + 2 \cdot \frac{6}{23} + \frac{1}{4} \right)
\]

6. (10 points) The grades on a Calculus II final are normally distributed with a mean of 145 and a standard deviation of 10. Write your answers as integrals and approximate the answers using calculators or tables.

(a) Find the probability of making a B if the range for B is 160 to 180.

\[
\int_{160}^{180} \frac{1}{10\sqrt{2\pi}} e^{-\frac{(x-145)^2}{200}} \, dx \approx 0.06657
\]

(b) Find the probability of failing the exam if a score below 120 is a failing grade.

\[
\int_{-\infty}^{120} \frac{1}{10\sqrt{2\pi}} e^{-\frac{(x-145)^2}{200}} \, dx \approx 0.00621
\]

7. (10 points) Find the limit of each sequence or state that it diverges.

(a) \( \lim_{n \to \infty} \frac{3n^3 - n^2 + 1}{2n^3 + 4n^2} = \lim_{n \to \infty} \frac{3 - \frac{1}{n} + \frac{1}{n^3}}{2 + \frac{4}{n}} = \frac{3}{2} \)

Hints: top, bottom by \( \frac{1}{n^3} \)

(b) \( \lim_{n \to \infty} (1 + \ln n)^{\frac{1}{n}} = \lim_{n \to \infty} e^{\frac{\ln \left( 1 + \ln n \right)}{n}} = e^{\lim_{n \to \infty} \frac{\ln \left( 1 + \ln n \right)}{n}} = e^0 = 1 \)
8. (30 points) Determine whether each series converges or diverges. State the test that you used and show that it applies.

(a) \[ \sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^n = (-1)^n + (0)^2 + \sum_{n=3}^{\infty} \left(1 - \frac{2}{n}\right)^n \quad \text{positive term series} \]

\[ \lim_{n \to \infty} \left(1 - \frac{2}{n}\right)^n = \lim_{n \to \infty} e^{\ln \left(1 - \frac{2}{n}\right)^n} = \lim_{n \to \infty} \left(1 - \frac{2}{n}\right)^n = e^{\lim_{n \to \infty} \frac{-2}{n}} = e^{-2} = 0 \]

So series diverges by test for divergence.

(b) \[ \sum_{n=1}^{\infty} \frac{n^2}{n^4 - n^2 - 1} = -1 + \sum_{n=2}^{\infty} \frac{n^2}{n^4 - n^2 - 1} \quad \text{positive term series} \]

Use limit comparison test comparing to \[ \sum_{n=2}^{\infty} \frac{1}{n^2} \] (a convergent \( p \)-series).

\[ \lim_{n \to \infty} \frac{\frac{n^2}{n^4 - n^2 - 1}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^4}{n^4 - n^2 - 1} = 1 \]

So both converge or both diverge by LCT. Since \[ \sum_{n=2}^{\infty} \frac{1}{n^2} \] converges, \[ \sum_{n=1}^{\infty} \frac{n^2}{n^4 - n^2 - 1} \] also converges.

(c) \[ \sum_{n=2}^{\infty} \frac{4^n}{n!} \]

Use ratio test.

\[ \lim_{n \to \infty} \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} = \lim_{n \to \infty} \frac{4}{n+1} = 0 < 1 \]

So series converges by ratio test.
9. (12 points) Use Taylor's formula to find the power series for \( \sin 2x \) centered at \( x = 0 \).

\[
\begin{align*}
f(x) &= \sin 2x & f(0) &= 0 \\
f'(x) &= 2 \cos 2x & f'(0) &= 2 = (-1)^{\text{even}} \cdot 2 \\
f''(x) &= -4 \sin 2x & f''(0) &= 0 \\
f'''(x) &= -8 \cos 2x & f'''(0) &= -8 = (-1)^{\text{odd}} \cdot 2^3 \\
f^{(4)}(x) &= 16 \sin 2x & f^{(4)}(0) &= 0 \\
f^{(5)}(0) &= 32 = (-1)^{\text{even}} \cdot 2^5
\end{align*}
\]

So Taylor's formula gives

\[
\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1} \cdot x^{2n+1}}{(2n+1)!}.
\]

Can also get (check) series by plugging \( 2x \) into series for \( \sin 2x \).

10. (12 points) Use known series to find the power series and interval of convergence for \( \frac{x^2}{1 - 2x^2} \) centered at \( x = 0 \).

This can be considered

\[
\frac{a}{1-r} \quad \text{for} \quad a = x^2, \quad r = 2x^2
\]

So we get

\[
\sum_{n=0}^{\infty} x^2 (2x^2)^n = \sum_{n=0}^{\infty} 2^n x^{2n+2}
\]

And we have interval of convergence \( |r| < 1 \) i.e.

\[
|2x^2| = 2x^2 < 1 \Rightarrow x^2 < \frac{1}{2}
\]

\[
\Rightarrow \left| x \right| < \frac{1}{\sqrt{2}}
\]
11. (12 points) Find the binomial series and radius of convergence for \( \frac{x}{\sqrt{1+x^2}} \).

Start with the binomial series for \( (1+x)^{-r} = \sum_{n=0}^{\infty} \binom{-r}{n} x^n \).

Substitute \( x^2 \) for \( x \) and multiply by \( x \) to get

\[
\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (x^2)^n \cdot x = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^{2n+1}.
\]

So converges for \( x^2 < 1 \) or \( |x| < 1 \) with \( \text{Rad} = 1 \).

\[
= x - \frac{1}{5} x^3 + \frac{\frac{3}{2}}{2} \cdot \frac{-\frac{1}{2}(-\frac{3}{2})}{2} x^5 + \frac{\frac{5}{2} \cdot \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})}{3!}}{3} x^7 + \ldots
\]

General term

\[
\frac{\frac{2n-3}{2} \cdot \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2}) \cdots (-\frac{2n-3}{2})}{(2n-3)!}}{(2n-3)!} x^{2n+1}
\]

\[
= x + \sum_{n=1}^{\infty} \frac{(-1)^n (1)(3)(5) \cdots (2n-3)}{5^n n!} x^{2n+1}
\]
12. (10 points) Bonus question.

(a) Find the power series for \( f(x) = xe^x \) using Taylor's formula.

\[
\begin{align*}
f'(x) &= e^x + xe^x \\
f''(x) &= e^x + e^x + xe^x \\
f'''(x) &= e^x + e^x + e^x + xe^x
\end{align*}
\]

\[
\begin{align*}
f(0) &= 0 \\
f'(0) &= 1 \\
f''(0) &= 2 \\
f'''(0) &= 3
\end{align*}
\]

\[
\sum_{n=0}^{\infty} \frac{nx^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n-1)!}
\]

(b) Differentiate the power series term by term.

\[
\sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n-1)!} = 1 + 2x + \frac{3x^2}{2!} + \ldots
\]

\[
\text{reindexing: } \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}
\]

(c) Differentiate \( f(x) \).

\[
f'(x) = e^x + xe^x
\]

(d) Find the power series for \( f'(x) \). You may use the answer from (a) and the known power series for \( e^x \).

\[
\begin{align*}
f'(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{nx^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}
\end{align*}
\]

(e) Compare your answers in parts (b) and (d). Are they the same?

\[
\text{Yes (after reindexing b.)}
\]